

$\Pr(j = k|x) = \frac{\exp(u_k(x))}{\sum_j \exp(u_j(x))}$  and  $V(x) = \log\left(\sum_j \exp(u_j(x))\right) + \gamma$  in  
a Static Discrete Choice Problem with T1EV Disturbance

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Suppose an agent has  $J$  choices, indexed by  $j \in \mathcal{J}$ . Conditional on the observable state of the agent denoted as  $x$ , suppose we know the functional form of the deterministic utility  $u_j(x), \forall j \in \mathcal{J}$ . The structural disturbances  $\varepsilon_j, j \in \mathcal{J}$  are additively separable to each decision and follow i.i.d. Type 1 Extreme Value (T1EV) distribution.<sup>1</sup> In other words, the agent solves

$$\max_{j \in \mathcal{J}} u_j(x) + \varepsilon_j$$

WLOG, let's calculate the probability of choosing choice  $a$  conditional on  $x$ . For simplicity, let's use the following notation:  $u_j(x) \equiv u_j$ ,  $(u_i - u_j) \equiv m_{ij}$ , and  $\varepsilon_a \equiv \varepsilon$  when appropriate. The conditional choice probability of choosing  $j \in \mathcal{J}$ ,  $\Pr(j = a|x)$ , is:

$$\begin{aligned} \Pr(j = a|x) &= \Pr\left([u_a + \varepsilon_a \geq u_j + \varepsilon_j], \forall j \in \mathcal{J} \setminus \{a\} | x\right) \text{ by definition} \\ &= \mathbb{E}_{\varepsilon_a} \left[ \Pr\left([u_a + \varepsilon_a \geq u_j + \varepsilon_j], \forall j \in \mathcal{J} \setminus \{a\} | x, \varepsilon_a\right) \right] \text{ by the law of iterated expectations} \\ &= \mathbb{E}_{\varepsilon_a} \left[ \prod_{j \in \mathcal{J} \setminus \{a\}} \Pr\left([u_a + \varepsilon_a \geq u_j + \varepsilon_j] | x, \varepsilon_a\right) \right] \quad \because \forall j, \varepsilon_j \text{ are i.i.d.} \\ &= \int_{-\infty}^{\infty} e^{-\sum_{j \in \mathcal{J} \setminus \{a\}} e^{-(m_{aj} + \varepsilon)}} \cdot g(\varepsilon) d\varepsilon \quad \because \text{Apply cdf for } \varepsilon_j, \forall j \in \mathcal{J} \setminus \{a\} \text{ and integrate } \varepsilon \text{ over the real line} \\ &= \int_{-\infty}^{\infty} e^{-\sum_{j \in \mathcal{J} \setminus \{a\}} e^{-(m_{aj} + \varepsilon)}} \cdot e^{-e^{-\varepsilon}} e^{-\varepsilon} d\varepsilon = \int_{-\infty}^{\infty} e^{(-e^{-\varepsilon})(1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}})} e^{-\varepsilon} d\varepsilon \end{aligned}$$

Change of variable:  $t = e^{-\varepsilon} \rightarrow dt = -e^{-\varepsilon} d\varepsilon$ . Then,  $\varepsilon \rightarrow -\infty \Rightarrow t \rightarrow \infty$  and  $\varepsilon \rightarrow \infty \Rightarrow t \rightarrow 0$ .

$$\begin{aligned} \int_{-\infty}^{\infty} e^{(-e^{-\varepsilon})(1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}})} e^{-\varepsilon} d\varepsilon &= \int_{\infty}^0 -e^{-t(1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}})} dt = \int_0^{\infty} e^{-t(1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}})} dt \\ &= \frac{e^{-t(1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}})} \Big|_{t=0}^{t=\infty}}{1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}}} = \frac{1}{1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-m_{aj}}} = \frac{1}{1 + \sum_{j \in \mathcal{J} \setminus \{a\}} e^{-(u_a - u_j)}} = \frac{\exp(u_a(x))}{\sum_{j \in \mathcal{J}} \exp(u_j(x))} \end{aligned}$$

So, we can finally say that  $\Pr(j = k|x) = \frac{\exp(u_k(x))}{\sum_{j \in \mathcal{J}} \exp(u_j(x))}$ .

Estimation of discrete choice models often relies on getting a closed-form relationship between the ex-ante expected utility and the choice probability conditional on the state variable. Let's compute the ex-ante expected utility from the discrete choice problem. In other words,

$$\begin{aligned} V(x) &= \int \max_{j \in \mathcal{J}} \{u_j(x) + \varepsilon_j\} g(\vec{\varepsilon}) d\vec{\varepsilon} = \sum_{j \in \mathcal{J}} \int (u_j + \varepsilon_j) \mathbf{1}\{u_j + \varepsilon_j \geq u_k + \varepsilon_k, \forall k \in \mathcal{J} \setminus \{j\}\} g(\vec{\varepsilon}) d\vec{\varepsilon} \\ &= \sum_{j \in \mathcal{J}} \int_{-\infty}^{\infty} (u_j + \varepsilon_j) e^{-e^{-\varepsilon_j}(1 + \sum_{k \in \mathcal{J} \setminus \{j\}} e^{-m_{jk}})} e^{-\varepsilon_j} d\varepsilon_j \end{aligned}$$

<sup>1</sup>If  $\varepsilon$  follows Type 1 Extreme Value distribution, the pdf is  $f(\varepsilon) = e^{-\varepsilon} e^{-e^{-\varepsilon}}$  and the cdf is  $F(\varepsilon) = e^{-e^{-\varepsilon}}$ .

Where the second line is exploiting the cdf form of iid T1EV structural errors across the decisions. Again WLOG, I calculate the first case,  $j = a$ , as the representative term of the summation. Denoting  $\varepsilon_a := \varepsilon$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} (u_a + \varepsilon_a) e^{-e^{-\varepsilon_a}(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} e^{-\varepsilon_a} d\varepsilon_a \\ &= \underbrace{\int_{-\infty}^{\infty} u_a e^{-e^{-\varepsilon}(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon}_{(1)} + \underbrace{\int_{-\infty}^{\infty} \varepsilon e^{-e^{-\varepsilon}(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon}_{(2)} \end{aligned}$$

Using the same change of variable,  $t = e^{-\varepsilon}$ , term (1) becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} u_a e^{(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}}) e^{-\varepsilon}} e^{-\varepsilon} d\varepsilon = u_a \int_0^{\infty} e^{-t(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} dt = u_a \frac{-e^{-t(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} \Big|_{t=0}^{t=\infty}}{1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}}} \\ &= \frac{u_a}{1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}}} = \frac{u_a}{1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-(u_a - u_k)}} = \frac{u_a \exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} \end{aligned}$$

Of course, noticing that we are taking expectation, we can simply say that (1) is

$$u_a \Pr\left([u_a + \varepsilon_a \geq u_j + \varepsilon_j], \forall j \in \mathcal{J} \setminus \{j\}\right) = \frac{u_a \exp(u_a)}{\sum_j \exp(u_j)}$$

We apply the same change of variable for (2) while realizing that  $t = e^{-\varepsilon} \Rightarrow -\log t = \varepsilon$ :

$$\begin{aligned} & \int_{-\infty}^{\infty} \varepsilon e^{(-e^{\varepsilon})(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} e^{-\varepsilon} d\varepsilon = \int_{\infty}^0 -(\log t) e^{-t(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} (-1) dt \\ &= \int_{\infty}^0 (\log t) e^{-t(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} dt \end{aligned}$$

Let's denote  $M \equiv (1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})$ :

$$\begin{aligned} & \int_{\infty}^0 (\log t) e^{-t(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} dt = \int_{\infty}^0 (\log tM - \log M) e^{-tM} dt \\ &= \int_0^{\infty} -\log(tM) e^{-tM} dt + \int_0^{\infty} \log M e^{-tM} dt \\ &= \int_0^{\infty} -\frac{1}{M} (\log y) e^{-y} dy - \frac{(\log M) e^{-tM}}{M} \Big|_0^{\infty} = \gamma \frac{1}{M} + \frac{\log M}{M} \\ &= \frac{\gamma}{1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}}} + \frac{\log(1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})}{1 + \sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}}} \\ &= \frac{\gamma \exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} + \frac{\exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} \log\left(\frac{\sum_{j \in \mathcal{J}} \exp(u_j)}{\exp(u_a)}\right) \\ &= \frac{\gamma \exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} + \frac{\exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} \log\left(\sum_{j \in \mathcal{J}} \exp(u_j)\right) - \frac{u_a \exp(u_a)}{\sum_{j \in \mathcal{J}} \exp(u_j)} \end{aligned}$$

We applied change of variable again in the second line:  $y = tM \Rightarrow dy = M dt$  and  $\gamma$  is Euler's constant. Note that the derivation of  $-\int_0^{\infty} (\log y) e^{-y} dy = \gamma$  involves  $\Gamma'(1) = \gamma$ . Knitting (1) and (2) together, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} (u_a + \varepsilon_a) e^{-e^{-\varepsilon_a}(1+\sum_{k \in \mathcal{J} \setminus \{a\}} e^{-m_{ak}})} e^{-\varepsilon_a} d\varepsilon_a \\ &= \underbrace{\frac{u_a \exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)}}_{(1)} + \underbrace{\frac{\gamma \exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)} + \frac{\exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \log\left(\sum_{k \in \mathcal{J}} \exp(u_k)\right)}_{(2)} - \frac{u_a \exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \\ &= \frac{\gamma \exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)} + \frac{\exp(u_a)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \log\left(\sum_{k \in \mathcal{J}} \exp(u_k)\right) \end{aligned}$$

Broadcasting this result across all  $j \in \mathcal{J}$ , we finally get  $V(x)$ :

$$V(x) = \sum_{j \in \mathcal{J}} \left( \frac{\exp(u_j)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \log \left( \sum_{k \in \mathcal{J}} \exp(u_k) \right) + \frac{\gamma \exp(u_j)}{\sum_{k \in \mathcal{J}} \exp(u_k)} \right) = \log \left( \sum_{j \in \mathcal{J}} \exp(u_j(x)) \right) + \gamma$$